THE

MATHEMATICAL GAZETTE.

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REPORT OF THE MATHEMATICAL ASSOCIATION COMMITTEE ON THE TEACHING OF MATHEMATICS IN PUBLIC AND SECONDARY SCHOOLS.*

SUMMARY OF RECOMMENDATIONS.

1. That a boy's educational course at school should fit him for citizenship in the broadest sense of the word; that, to this end, the moral, literary, scientific (including mathematical), physical and aesthetic sides of his nature must be developed. That in so far as mathematics are concerned, his education should enable him not only to apply his mathematics to practical affairs, but also to have some appreciation of those greater problems of the world, the solution of which depends on mathematics and science.

2. That the utilitarian aspect and application of Mathematics should receive a due share of attention in the mathematical course.

3. That the mathematical course in the earlier stages should not be concerned exclusively with Arithmetic, Algebra, and Geometry; that such subjects as Trigonometry, Mechanics, and the Calculus, should be begun sooner than is now customary and developed through the greater part of the boy's school career, so as to give him time to assimilate them thoroughly, and enable him to cover more rapidly, at a later stage, the higher parts of Arithmetic, Algebra and Geometry.

4. That no boy should leave school entirely ignorant of Applied Mathematics (*e.g.* Mathematics relating to the Stability of Structures, Motion of Bodies, Electrical Plant, Astronomy, etc.).

5. That, while the average boy should receive careful and adequate instruction, the boy of high talent should receive special attention, as the value to the race of carefully trained superior talents is incalculable. A well-equipped secondary school should be so staffed as to be able to educate *pro viribus* both the average boy and the boy of genius.

6. That a boy who takes Mathematics as his main subject in the later part of his school life should also, in general, study Science, as well as carry on some form of literary study; and that the general

* Throughout this Report the word *boy* is to be taken as referring to pupils of either sex.

educational purpose underlying the choice of these various subjects should be made manifest to the boy.

7. That the time devoted to Mathematics in the Secondary Schools before the period of specialisation should be at least six periods per week (excluding time for preparation at home).

8. That the teaching and organisation of Mathematics and Physics should be in the closest possible co-ordination.

9. That every teacher of Mathematics should go through (1) a course of *mathematical training* at the University to be followed by (2) a course of *professional training* in the Theory and Practice of Teaching with special reference to Mathematics, at a Training College, the two courses not to run concurrently.

10. That the mathematical teacher should also receive training of a less intensive kind in some subject in which his Mathematics can be definitely applied, *e.g.* Geography, Physics, Chemistry, Engineering, Manual Training, Astronomy, etc.

11. That teachers of insufficient ability or knowledge should not be promoted to be Heads of Departments simply on the ground of long and faithful service. Long and faithful service deserves recognition, but in some other way. The Head of a Mathematical Department has to teach the advanced work—which requires knowledge—and to draw up the general school syllabus—which requires outlook.

12. That Heads of Mathematical Departments and Specialist Teachers of the higher branches of Mathematics in the Advanced Departments of the Secondary Schools should have tolerably short routines in order that they may be able to read more widely in their subject and to study its modern development (cf. 15). Only thus can the knowledge imparted to the boys be kept up to date.

13. That an External Examination Syllabus should be frequently revised by a joint body consisting of representatives of the teachers themselves and of the external examining body. Otherwise the syllabus, being stereotyped, tends to become obsolete, and teachers have to teach what they have ceased to value.

14. That it is more important to TEACH boys than to EXAMINE them; that the number of examinations at present conducted in the majority of schools should be reduced, inasmuch as the setting of examination papers consumes the master's time and energy, thereby lowering his teaching capacity, while the continuous pressure of working at examination papers for days on end is an unproductive strain on the boys.

15. That every Secondary School should be provided with a Mathematical Library, containing books of a more general character than the ordinary text-books, in order that the pupils and masters may be enabled to widen their horizon and catch a glimpse of the regions beyond.

16. That portraits of the great mathematicians should be hung in the mathematical class-rooms, and that reference to their lives and investigations should frequently be made by the teacher in his lessons, some explanation being given of the effect of mathematical discoveries on the progress of civilisation.

I. INTRODUCTORY.

The object of the present Report is to discuss on a broad basis the Teaching of Mathematics in Public and Secondary Schools. The Mathematical Association has not for many years issued a report of this character, certainly not during the present century. Its recent

reports have been addressed to the Teacher and the Examiner. They have been concerned mainly with questions of detail. Discussions of root-principles have been avoided, possibly on account of the difficulty of coming to an agreement. The present report is of a more general character. It is hoped that such a report will prove useful not only to the mathematical teacher and examiner but also to the general public, and in particular to the non-mathematical element among those directly or indirectly concerned with the organisation of education.

II. OBJECT OF MATHEMATICS IN A LIBERAL EDUCATION.

One of the objects of the education provided in a modern Secondary School is to give a broad general training, such as will enable a man to fulfil to best advantage his functions as a member of the community. After leaving school he will learn the technicalities of some trade, profession or vocation, and they will be expected to use their school education as the broad basis on which to build this post-school structure of technical training. But man's whole life ought not to be centred merely on gaining a livelihood. He must have recreation, and he must be able to understand—however imperfectly—some of the problems presented by the world around him. At school the pupil should receive a training that will develop as far as possible *all* his intellectual powers, and not merely those required for the pursuit of his particular trade or calling. A pupil's training at school will therefore fall roughly into five departments: (1) moral, (2) literary, (3) scientific, (4) physical, (5) aesthetic. With the effects of the moral, literary, physical and aesthetic parts of his education we are not called upon to deal in detail in this report, but no report on mathematical teaching can be regarded as satisfactory which does not recognise that no boy can be considered as having a good education unless his literary and aesthetic faculties have received proper training, and that such training cannot be replaced by mathematical and scientific instruction however thorough.

An educated man should be able to appreciate literature, to study the development of his own and other nations, to understand political, municipal and civic problems, and to enjoy art and music. On the other hand there are vast tracts of human thought and activity which require a mathematical and scientific training, if one is to be in a position to appreciate them even remotely.

Thus, if we take a wide view of human life in relation to mathematics and science, we see that the material side of modern civilisation stands on a foundation of applied mathematics. This statement will not be controverted by anyone familiar with the work of modern scientists and engineers, and with the part played by mathematics in their achievements. It is true that the average man takes no direct part in these developments, but it is not fitting that he should live as a mere parasite on the organisation that keeps him alive. It is the work of the few to render navigation secure by predicting years ahead the motions of the heavenly bodies, to link the continents together with steamship lines and electric cables, to cover new corn-bearing lands with railways, to tunnel the Alps and the Andes, to prepare for the day when our coal will fail by extracting electric power from waterfalls and tides, to renew the fertility of the land by materials derived from the air we breathe. These are some of the contributions of science to our means of existence; and mathematics is the tool that science uses. The modern man should have at least some conception of the means by which these results, so vital to him, are obtained.

We thus see that school mathematics and science have both a

"utility" and an "outlook" value. From a purely utilitarian standpoint, mathematics teaches the ordinary man how to calculate what he ought to pay in taxes and so forth, but the "outlook" aspect of mathematics is no less valuable though its effects are more abstract and require a greater effort of faith. We assume that the majority of boys will not make any extensive or profound use of mathematics in after life, that they will not even be able to follow in detail the mathematical methods of engineering and applied science. But they can be so taught that a vista is opened up to them through which they may see the tremendous potentialities of the study whose elements they are mastering. They should be brought to the stage from which a broad general view may be obtained over the country of applied mathematics; and they may be shown the beginnings of a few of the roads that lead through this country. A public must be created able to realise what science and mathematics are doing for the world, and to form some general conception of the means employed.

The average man will not be more than a spectator of the world's material progress; we have suggested that he should be an intelligent spectator. But the world needs an increasing number of workers trained to use mathematics; there must be specialists. In analysing the mathematical outlook at a secondary school, one is too apt to direct one's attention exclusively to the training of the average boy, who will never be a brilliant mathematician. But boys who are destined to become good and sometimes great mathematicians do exist in the schools. These boys ought to be trained to the highest level of their capacity. Sir William Ramsay, in his essay on Lord Kelvin, says, "It is precisely those boys, who are unique and unlike their fellows, who are of value to the race, and every chance should be given to exceptional talent." It is often said that good mathematicians can be left to look after themselves. There never was a greater mistake. If left to themselves they are merely trying to traverse in blind, ignorant and amateurish fashion the path explored by the great pioneers who have gone before them. When a path through the jungle has been explored, maps are immediately constructed so that those who follow may not be put to useless and unnecessary labour. So it is in science and mathematics. The good mathematicians in the schools are few in number, but they are the potential leaders of the future in those departments of human activity which rest on a mathematical basis. The educational equipment of either school or country must therefore be regarded as deficient unless proper specialised instruction is provided for the abler boys. This does not mean that the average boy is to be neglected. A staff must exist at any public or secondary school capable of giving both to the brilliant boy and to the rank and file suitable and adequate instruction.

It will be seen that this Committee is far from advocating that any subject of the school curriculum should be taught to the exclusion of all others. Both the scientific and the literary power of appreciation must be trained, and they can to a certain point be trained side by side. Every teacher of mathematics can be, and should be, in a certain sense a teacher of English as well. *Direct* teaching of English should enter into the teaching of every boy; but *direct* teaching of English need not supplant *indirect* teaching. There must be special hours devoted to English, but English must still be taught incidentally at all other lessons given in the mother tongue. Different subjects will bring out different features of the use of English; and the special feature that mathematics and science can emphasize is clearness and precision of language. Sufficient has been said to show that literary and scientific subjects are in this report regarded as complementary and not antagonistic.

In trying to judge whether a boy's mathematical training at his secondary school has been on sound lines, we must avoid the attitude of mind of the country grocer who condemned the whole system of English education on the ground that his new apprentice from the local grammar school, when faced with his first customer, did not know the difference between a florin and half-a-crown. A liberal education does not seek only to furnish a boy's mind with facts, however useful, but to teach him principles in the light of which he may appreciate and master whatever phenomena may present themselves to him.

III. THE EFFECT OF MATHEMATICS ON THE PUPIL'S MIND.

In reply to the memorandum of the five humanist associations, the Mathematical Association expressed the view that:—"From a school course of mathematics the pupil should acquire valuable habits of precise thought and expression." To confirm this aspiration we have Mr. Nowell Smith's opinion. "These twin engines of education, classics and mathematics, are adapted, partly by long practice, but partly as I do believe, by their very nature, to discipline the youthful mind to habits of intellectual honesty, of accuracy, of industry and perseverance." *

The psychology of education is still in the making; till we know more of this science it is wiser to build on the commonsense ground of everyday experience than on far-reaching theoretical considerations. We are still fairly ignorant of the effect on the mind of any particular course of diet; till we know a good deal more our safest guide will be appetite. Now the idea that mathematics is the most repulsive of all subjects is so firmly embedded in popular belief that it may seem hopeless to base any argument on appetite. But the experience of good teachers does not by any means confirm popular belief. The experience of good teachers is that a small minority of boys are almost incapable of mathematical reasoning, and derive no perceptible benefit from it; that another small minority take to the subject quite naturally and are happy in no other element; and that the great majority of boys can be led to find mathematics an interesting study, and to face a fair amount of solid work without grumbling. This is the modern experience of mathematical teaching. Add to this a respectable record of antiquity, going back presumably to the time of Pythagoras, and the inference is reasonable that mathematics is an instrument of education found to be in conformity with the needs of the human mind.

We need not hesitate to put the pleasure motive among the foremost. "Perhaps some kinds of study would have fared better if their defenders had dwelt more upon the pleasure they afford and less upon their supposed utility." † It would be surprising if mathematical ideas, properly presented, failed to give pleasure to the average mind. The quantitative treatment of any subject is mathematics, and mathematics is at the root of all things measurable. "Nature herself seems to have a taste for mathematical methods." ‡ A mode of thought so fundamental might be expected to arouse a sympathetic vibration in the intellect.

Boys are sent to school to make intellectual efforts. There is no difficulty is setting tasks, in any subject, that call for effort. But effort is not all that is needed. Effort must lead to success or else to discouragement and apathy. Again, it is good that the boy should be

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* *Cambridge Essays*, p. 115.

† Bryce, *Cambridge Essays on Education*, p. xiv. Edited by Dr. A. C. Benson, Cam. Univ. Press, 1917.

‡ Inge, *Cambridge Essays on Education*, p. 21.

able to judge of his own success. Now in these two respects mathematics has advantages. The tasks can be graduated nicely to the powers of the worker; there is never any necessity to set hopeless tasks. At times, of course, the teacher may deliberately set too hard a problem, perhaps to lead up to a new method which makes the problem possible; or perhaps to correct intellectual pride. But this is an advanced touch, and ordinarily if the work is too hard for the boy the fault is with the teacher, not with the subject.

Then, as to the boy judging his own success, there is a great difference between one subject and another. There is no wish here to depreciate literary studies; they can never be replaced by anything that mathematics or science has to offer. But, speaking generally, in literary work, the verdict is a matter of taste and authority, the matured taste of the teacher or the authority of the classical author. The boy as he develops will learn, no doubt, to appreciate the criticisms offered, and the teacher will be successful in proportion as he gains the boy's assent; but the boy can hardly be his own judge. In mathematics, to a great extent, he can judge his own work. In problems of calculation he can be taught to apply checks to his work, which will tell him plainly if he has gone wrong. In geometry the teacher can say, "This step is wrong, why?" and the boy can satisfy himself that the step is wrong, and why it is wrong. In this way mathematical work is well fitted to develop an independent and self-reliant habit of mind.

It is needless to insist on the pedagogic value of the precision which characterises mathematics; this has always been accepted as a peculiar merit of the subject. It is true that this merit is claimed by most teachers for their own special subject; and this reflection should perhaps lead us to refrain from excessive reliance on an argument that has become associated with a rather meagre educational creed. But it may fairly be said that there is no room in mathematics for vagueness of thought, the hiding-place of the lazy mind: the boy must stand up to his difficulties; there is no escape for him.

How far the habit of precise thought in mathematics begets a habit of precise thought in other matters is a question for the psychologists. The mathematical specialist lives in a world of thought where most things are clear-cut, and most questions have their "Yes" or "No." Is the mind so trained well fitted to deal with practical problems of life, where things refuse to be treated too absolutely? Is not the student of mathematics in need of a corrective to his too-great precision, such a corrective as History, which calls into play judgment of men and balancing of probabilities?

Teachers of elementary work need not be moved by these speculations: they find themselves combating vagueness all day long, whatever subject they are teaching, and they accept mathematics as an ally.

IV. THE NATURE OF MATHEMATICAL REASONING.

Mathematics gives scope for inductive as well as deductive thought. Geometry in some of its aspects has many of the characteristics of a physical science, and lends itself largely to inductive processes; opportunities for induction, intuition and imagination are among the essential merits of mathematical study.

The logical process of induction is found in Mathematics as in other spheres of work where the mind acts normally in face of difficulties. The justifiable claim of the mathematical savant for rigid proof is apt to mislead uninstructed public opinion by the 'defect of its quality,' and to arouse a prejudice which is an additional handicap in teaching. Such a view was expressed by Huxley as follows:

"Mathematical training is almost purely deductive. The mathematician starts with a few simple propositions the proof of which is so obvious that they are called self-evident, and the rest of his work consists of subtle deductions from them."* And on another occasion he remarked that mathematics "is that study which knows nothing of observation, nothing of induction, nothing of experiment, nothing of causation."†

This conception of mathematics as purely deductive was rejected by Sylvester in his famous British Association address at Exeter in 1869. He affirmed that mathematical analysis "is unceasingly calling forth the faculties of observation and comparison, that one of its principal weapons is induction, that it has frequent recourse to experimental trial and verification, and that it affords a boundless scope for the exercise of the highest efforts of imagination and invention."

"Most, if not all, of the great ideas of the modern mathematicians have had their origin in observation. Take, for instance, the arithmetical theory of forms, of which the foundation was laid in the Diophantine theorems of Fermat, left without proof by their author, which resisted all the efforts of the myriad-minded Euler to reduce to demonstration, and only yielded up their cause of being when turned over in the blowpipe flame of Gauss's transcendent genius; or the doctrine of double periodicity, which resulted from the observation by Jacobi of a purely analytical fact of transformation; or Legendre's law of reciprocity or Sturm's theorem about the roots of equations, which, as he informed me with his own lips, stared him in the face in the midst of some mechanical investigations connected with the motions of compound pendulums; or Huyghens' method of continued fractions to which he appears to have been led by the construction of his Planetary Automaton."

Other eminent mathematicians have spoken to the same effect. Thus we find Weierstrass saying: "It is a matter of course that every road must be open to the searcher as long as he seeks; it is only a question of the systematic demonstration" (*Werke*, vol. ii. p. 235). Or we might refer to Professor Klein, who required the fullest logical treatment to be accompanied by an equal development of intuitive representation.

Much more might be quoted in the same sense, but we add one short note, also from the pen of Professor Klein. "We are living in a critical period, similar to that of Euclid."

Though this modern critical movement requires neither our endorsement nor our patronage, it does require much thoughtful consideration from teachers. We must prevent the suppression in modern youth of that naïve intuition which according to Professor Klein was "especially active during the period of the genesis of the differential and integral calculus. Thus we see that Newton assumes without hesitation the existence, in every case, of a velocity in a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivatives." Nor would it be difficult for teachers to reach a satisfactory arrangement for work, if experts in criticism were alone to be consulted. Thus we find in Russell: "What I do wish to maintain . . . is that insight, untested and unsupported, is an insufficient guarantee of truth, in spite of the fact that much of the most important truth is first suggested by its means."‡ The classical instance of a brilliant induction established by the painstaking examination of individual cases is Newton's discovery of the Binomial Theorem for a fractional index, a deductive proof being given later on by Euler.

* *Macmillan's Magazine*, June, 1868. † *Fortnightly Review*, 1868.
‡ *Our Knowledge of the External World*, p. 20. (Open Court 1914).

The basis of such an agreement is to be found in the original records of the development of mathematics as revealed by historical research. The study of real objects, the growth of feeling through real sensory perception, through the eye and the sense of touch, etc., the recognition of sameness and similarity, the discrimination of difference, all these leading to generalisation by induction—such is the kind of work that must precede any systematic study arranged by logical deduction from a given number of abstract premises taken as exact. Such a critical deductive study must be confined to the mathematical specialist course, primarily at the universities, and extended only with caution to school teaching. It may be asked—Is there no place for deductive work in the general or non-specialist course? Certainly there is, but in its natural place. The number of unrelated facts that can be accumulated and retained by the mind is not great, and both induction and deduction will be used as required, for the purpose of suggesting relations and criticising or generalising any suggestions put forward. When the case is once grasped by young minds, the difficulty is not usually in making suitable deductions, but in grasping the essential points in a sufficiently abstract form. This difficulty is very similar to that experienced by the university graduate facing a problem in a modern workshop: the question is not stated explicitly as in an examination paper; the expert must seek his facts, select by his own judgment those essential to his case, and then supply the answer. In such work, observation and trained instinct are as necessary and as useful as capacity for logical deduction. There is no wish, however, to depreciate the latter, and still less to discourage the student from acquiring profound knowledge. It was the immortal Faraday who declared: "The world little knows how many of the thoughts and theories which have passed through the mind of a scientific investigator have been crushed in silence and secrecy by his own severe criticism and adverse examination; that in the most successful instance not a tenth of the suggestions, the hopes, the wishes, the preliminary conclusions have been realised." For the mathematician this necessary process of sifting is primarily one of trained capacity in deductive logic. And it was Pasteur, equally a master in inductive science, who said: Dans les champs de l'observation le hasard ne favorise que les esprits préparés."

V. THE HISTORY OF MATHEMATICAL TEACHING WITH SPECIAL REFERENCE TO ENGLAND.

Mathematics was not generally included in the Secondary Schools curriculum in England before the nineteenth century. It is a confluence of three streams flowing from widely different sources, whose waters to this day are not completely mingled. One stream is that of *commercial arithmetic*. This art, known as algorism, found its way into Italy during the thirteenth century from Arabic and Indian sources. From Italy it spread through Europe by the channels of commerce. It was never much concerned with scientific proof, and finally, in the English Schools for writing and arithmetic, it became frankly empirical. Early in the nineteenth century it was probably taught by the writing master, if taught at all. As the century wore on, arithmetic was incorporated with mathematics; it began to be taught in a reasoned manner and not merely as a collection of rules. Some of the rules were dropped, as connected with obsolete business conditions. As processes of manufacture and engineering became more scientific, the subject matter of arithmetic became less purely monetary and commercial, and more industrial; the interest of school problems

was no longer mainly how to buy and sell things, but to an increasing extent how to make things. In fact school arithmetic still shows clear signs of its commercial origin.

Greek arithmetic, as transmitted to the medieval universities by Boethius, entirely divorced from practice and computation, was concerned chiefly with the properties, mystical and other, of numbers. The chapter on prime numbers and factors in our modern arithmetics is the only vestige remaining of this Boethian arithmetic.

The second stream is that of *geometry*. From empirical beginnings, possibly in Egypt, this science reached a high degree of academical abstractness in Greece and the Grecian world.

The third stream is that of *algebra*, springing from Indian origins through Arabic transmission and first developing in Europe at the close of the Middle Ages.

The Greeks made little progress in algebra.

These converging streams gradually made their way to England and influenced in diverse ways the rudimentary teaching of mathematics in this country.

Presumably arithmetic forced its way into public schools upward from the commercial schools, while geometry and algebra came down from the universities. The diverse origins of the three subjects are shown clearly enough by the arrangement of modern books, and by the restrictions, now almost obsolete, which used to forbid the blending of the subjects; algebraical symbols were not to be used in arithmetic and geometry, and so forth.

No exhaustive and authoritative work has yet been written giving in full detail the history of the teaching of Mathematics in the Schools of England. Many interesting facts are known, however, regarding the struggles made by the subject to obtain a proper status and footing in isolated schools at various periods of English history, and it may therefore be of both interest and value to record some of these.

The first two schools to which we shall refer have a special interest at the present day, when the whole world has come to realise the vital contribution made to the maintenance of civilisation by the British Navy. The Mathematical School at Rochester, and the Mathematical Department at Christ's Hospital were both founded in the days of Charles II. definitely and explicitly to provide a proper mathematical education for boys who looked forward to entering the sea service or the Royal Navy. Sir Joseph Williamson, the founder of the Rochester Mathematical School, was the son of the Rector of Bridekirk in Cumberland. He was educated at Westminster and later on became Fellow and Tutor of Queen's College, Oxford. When he was twenty-seven, he gave up his university career for diplomacy and ultimately married Lady Catherine O'Brien, Baroness Clifton. Thereafter he withdrew largely from public life and devoted himself to the management of his wife's Cobham estate, which led to his connection with Rochester. In his will, when founding his Mathematical School, he says: "Item.—I do give and bequeath Five thousand Pounds to be laid out by my Executors hereinafter named in the purchasing some Lands and Tenements in England for and towards the building, perfecting, carrying on and perpetually maintaining of a free School at Rochester and of a Schoole Master or Schoole Masters for the Instruction and Education of such youth there who were or shall be Sons of Freemen towards the Mathematicks, and all other things which may fitt and encourage them for the Sea Service or Arts and callings leading or relating thereto."

The following is a sketch (by Mr. W. W. Rouse-Ball) of the Mathematical Department of Christ's Hospital in the days of Charles II.

A School Course in Mathematics in the Seventeenth Century.

The History of the Teaching of Elementary Geometry has recently formed the subject of an interesting work published in America by Dr. A. W. Stamper. In it, in one place, I find a statement that there is no mention of Geometry being taught in England outside the Universities during the sixteenth and seventeenth centuries, and in another place it is said that it may be safely guessed that the place of Euclid amongst English school-books dates at the earliest only from the middle of the eighteenth century. Dr. Gow, the Headmaster of Westminster, also makes the latter remark in his *History of Greek Mathematics*, and adds that he can find no useful information on the mathematical curriculum of an English public school before 1750.

We have, however, some information about the mathematical teaching at one English public school—Christ's Hospital, London—in the seventeenth century, which may be worth giving in detail, though it should be stated at once that the circumstances there were peculiar, and it is not typical of the work at a normal public school at that time.

At that school there were special provisions, on the foundation of Charles II., for teaching mathematics and the elements of navigation to boys who intended to enter the Royal Navy. The King, after establishing this department, directed Sir Jonas Moore to compile a text-book for the boys. Moore died before he had finished it, but it was completed by his sons-in-law, W. Hanway and J. Potenger, with the aid of Mr. Perkins, then the chief mathematical master in the school, Flamsteed and Halley. The whole was published in 1681, and contains sections on Arithmetic, Algebra, Practical Geometry, together with the substance of Euclid's *Elements*, i.-vi., xi., xii., Plane and Spherical Trigonometry, Cosmography, Navigation, the Sphere, Astronomical and Logarithmic Tables, and Geography.

The work had a considerable reputation, and is excellent of its kind, but we may infer from the correspondence described below that, in fact, it was not used in the school. This may have been due to the death of Mr. Perkins in 1680 or 1681. A few years later the question of the school course in mathematics, admittedly insufficient, was raised, and it was determined at the same time, taking advantage of the opportunity, to connect or combine with the King's foundation another one in the school, due to the liberality of a Mr. Stone. The new scheme, with a statement explaining what change is introduced, was sent to Isaac Newton, and his criticisms thereon invited. A draft of his reply and some of his memoranda on the subject are preserved among the Portsmouth papers.

The syllabus previously in use had not been preserved, but from the extant papers it would seem that at least it required the boys to study the earlier parts of the first book of Euclid's *Elements*, the tenth, eleventh, and twelfth propositions of the sixth book; and learn arithmetic. The course lasted two years. Perhaps this represented all the theoretical mathematics then normally prescribed to the King's Scholars, for Newton notes the following omissions. There was no "symbolic arithmetic," *i.e.* no use was made of algebraical symbols and methods. There was no "taking of heights and distances and measuring of planes and solids."

There was nothing "of spherical trigonometry, though this was requisite for many problems in astronomy, geography and navigation." Nothing was taught of "Mercator's chart" or "of computing the way of a ship," or "of longitude, amplitude azimuths, and variations of the compass." And, lastly, there was no "word of reasoning about force and motion, though it be the very life and soul of mechanical skill and manual operations." He continues, "by these defects it's plain that the old scheme wants not only methodizing, but also an enlargement of the learning." Of course the specific criticisms were made with reference to the fact that the boys looked to a career in connection with the sea.

The scheme drawn up to replace this, and submitted to Newton, was designed to occupy two years and a half. He at once wrote cordially approving the plan of working the two foundations in conjunction. He said he thought it would be desirable to keep the number of boys in the King's School constant by promoting to it boys from Stone's foundation as vacancies occurred, but pointed out that any way of combining the two foundations must be tested by the results, though experience shows "that children thrive best in great schools where there are a competent number of boys in every form to create an emulation and whet one another's parts." It is unnecessary to describe this scheme at length, because on the advice of Newton it was considerably modified and extended by the addition of mechanics.

Of the final scheme there are various drafts, differing in minute details. I think we may take the following as being substantially that ultimately approved for the abler boys. The subjects of study were to be Arithmetic; Algebra; Plane and Solid Geometry, practical with rule and compass as well as theoretical; Plane Trigonometry; Drawing and Designing; Instruments and their use; Cosmography, including therein the rudiments of Astronomy and the art of making maps and charts; the use of spherical triangles; and Mechanics. Also the applications of the above to sea problems, laying down courses, and determining positions. Some schedules of the extent to which the subjects should be read are given, but perhaps the above description will suffice.

Newton's letter, sent with the final draft embodying his criticisms, covers eight or more closely written foolscap pages. The general line of his comments is as follows. He begins by a detailed examination of the scheme submitted to him, and discusses the order in which the divisions of the different subjects should be taught. Apart from defects here indicated, he objects to the limitations as encouraging working by rule of thumb, for thus it "comprehends little more than what a child may easily learn by imitation . . . without understanding . . . the reason of what he does," and puts "an industrious blockhead on the same level as a child of parts." Moreover it does not assist a boy "in inventing new things and practices . . . or in judging what comes before him," which should be the aim of education. He considers, however, that, with his amendments and with the addition of mechanics, the new scheme would be satisfactory.

Next he emphasizes the great importance of teaching theoretical mechanics to the boys. No doubt, he says,

practical mechanics is also necessary, but the principles of the subject must be understood if one is to be able to deal with new problems. He asserts that the French engineers were superior to those in the British service, and that this was due to the fact that training in France included theoretical mechanics, though since the schools in that country contained a mixture of all sorts of capacities he believed that lads turned out from Christ's Hospital, which draws "children of the best part selected out of a great multitude," should excel them. He raises the question whether the addition of mechanics will take up too much time, and says that the addition of six months to the two years and a half allowed for the course submitted to him would permit its inclusion, and adds that it may be more for the boys' advantage to spend this half year at school in an important part of learning which they cannot get at sea, than at sea in learning what they will afterwards learn more readily if well instructed at school before they go thither, though, if the extra time cannot be allowed, he thinks that, perhaps, they may run through all parts of it in two years and a half, but not so well. "And I would advise that they should rather be allowed three full years than be sent away smatterers in their learning."

He expresses opinions that in any case it was important that the masters should be able to reckon on a definite period for which the school course would last, and that boys ought not to be allowed to leave their courses incomplete whenever vacancies occurred for which they were eligible; that the school course should be terminated by a school examination, but that the results of private examinations should be regarded by the Governors as confidential; and that it was desirable that boys who had left the school should be encouraged to keep in touch with it by sending to it accounts of their observations, etc.

As far as text-books are concerned, he approves the use of the *Synopsis Algebraica* (no doubt that compiled by J. Alexander, 1693), Ward's *Trigonometry*, and Euclid's *Nova Methodo*, which had been suggested to him, though as to the last-named work he would prefer that the boys should read Euclid in the original, and he suggests that at any rate the abler boys should read, in Barrow's English translation, books i.-vi., xi., xii. As for the doctrine of the sphere he recommends the use in the school of the first book of Mercator's *Astronomy*. He sees no disadvantage, indeed rather the reverse, in using books written in Latin.

At Christ's Hospital the boys were receiving a special training, so the mathematical course was taken to a higher standard than at other public schools at that time. This must be borne in mind, and there can be no doubt that the mathematical work at public schools was usually far less than that here described, but I think as a definite detailed example of a school course at the time it is worthy of record, and that Newton's comments on it are of interest.

W. W. Rouse Ball.

Trinity College, Cambridge, *January*, 1910.

The above description of the Mathematical Department of a great school in the seventeenth century states an ideal which is rarely grasped and probably never achieved in practice even at the present

day. The history of the Mathematical Departments of other great schools is a record of tinkering, makeshift and intermittent struggle for the recognition and status of mathematics alongside the classical studies, until we come to the present generation. The following are brief notes on the history of mathematical teaching at one or two of the great schools. In the case of Harrow, Cajori (*History of Elementary Mathematics*, page 207) states that "vulgar fractions, Euclid, geography and modern history were first studied in 1829." Before the Public Schools Commission of 1861 Marillier read the following: "In 1819, the late Dean of Peterborough, at that time Headmaster of Harrow, seeing the want of more mathematics in the school, the sixth form alone reading Euclid once a week with him, and the remainder of the boys being left to take private mathematical lessons of my predecessor, who was then very aged, invited me to undertake private mathematical instruction of such pupils as should desire it; and I accepted the appointment." In 1837 the whole school was arranged for mathematical classes, and that was the date at which mathematics seems first to have become a compulsory subject throughout the school.

At the Merchant Taylors' School "mathematics, writing and arithmetic were added in 1829." At Eton mathematics was not compulsory till 1851. As regards Winchester, Canon Fearon says that "in 1833 a school prize was started by Sir Wm. Heathcote, jointly for Classics and Mathematics. The founder's rules for his prizes enact that the examination in mathematics will be conducted wholly on paper, and will commence with questions in arithmetic (to decimal fractions inclusive), and in the first Book of Euclid, in which every candidate will be expected to *have prepared himself*, followed by a series of questions increasing in difficulty, until the extreme limits of the acquirements of each candidate shall be ascertained." In the following year, 1834, the first mathematical master at Winchester, Walford, was appointed presumably in order that the boys might not be compelled to go on "preparing themselves." Till 1861 he was the only mathematical master for the whole school (then 150 or so) except for a writing clerk who cleaned slates, etc. Canon Wilson says "at Rugby the first mathematical master was Robert Bickersteth Mayor, 3rd. wrangler in 1842, appointed April 1845. Up to that time arithmetic was taught by writing masters, Jack Sale and Pooley, who continued into my time, 1859. Mayor soon got an assistant, R. B. Smythies, not at first recognised as one of the staff, but who was allowed to take a house. Highton, afterwards Headmaster of Cheltenham, came as Science master and taught mathematics also. I was his successor in both." It would be interesting if more information were collected as to the early days of public school mathematics. Possibly the public schools were behind the grammar schools.

As an example of mathematical teaching in a purely private school, the following is of interest.

In the *Vulgar Arithmetique* of Noah Bridges, a Putney Schoolmaster—1653—there is an advertisement of his school announcing that he teaches Greek, Latin, Writing, Arithmetic, Geometry, Algebra and Trigonometry.

Judging from the contents of Bridges' books, he was a man of very independent character, and therefore his school was probably not typical of private schools of his time any more than the mathematical department of Christ's Hospital was typical of Public Schools, but his syllabus of subjects is nevertheless worth noting.

No recorded statement appears of the arguments urged in favour of the introduction of mathematical teaching, but de Morgan quotes Edmund Wells (Young Gentlemen's Course in Mathematics, London 1714): "Young gentlemen must not be so brisk and airy, as to think,

that the knowing how to cast Accompt is requisite only for such Undertakings as shop-keepers or Tradesmen," and for the sake of taking care of themselves "no Gentleman ought to think arithmetic below Him that do's not think an Estate below Him."

If one studies closely the development of mathematical teaching from earliest days, one finds that only at the present day has the subject ceased to be taught to the generality of pupils as a branch of abstract logic. It was thus that Euclid conceived of Geometry when he wrote his famous *Elements*. If one questions well-educated men living to-day, of ages ranging from fifty to seventy or over, one finds, generally speaking, that they regard the mathematics of their school and university days as something entirely divorced from the world around them. The cleverer boys profited by the mathematical instruction they received, while the dullards endeavoured to get up their geometrical propositions by heart. Still more remarkable was the spectacle for more than a thousand years of both teacher and taught wrestling with such propositions as the first in Euclid, Book VI., on "the areas of triangles between the same parallels bearing to one another the ratio of their bases," the pupil trying to stimulate his faltering memory while the teacher has the strategic advantage of the open book before him, but neither understanding the profound logical subtleties to which they are rendering such strenuous lip-service. The dawn of modern times witnessed the revolt of some more enterprising teachers against this comedy, and in 1871 was founded "*The Association for the Improvement of Geometrical Teaching.*"

The result of this was the preparation and publication of suggestions for improved methods of teaching geometry. It was inevitable that the Association should widen the scope of its labours, and in due course it altered its name to "The Mathematical Association" and issued syllabuses dealing with improvements in the teaching of Arithmetic, Algebra, and Mechanics as well as Geometry. Finally, it has yet again widened the basis of its efforts and is at present preparing suggested syllabuses of instruction on the "Mathematics of the Specialized Occupations," such as Mining, Commerce, Navigation, Engineering, Agriculture and so forth. A meeting consisting of a General Section and an Advanced Section is held once a year, at which teachers can mutually aid one another by discussion, lecture and informal talk. At the General Section the teaching of mathematics to the average boy is kept in view, while in the Advanced Section lectures by leading mathematicians on their own branches of study are arranged in so far as these bear on the teaching of higher mathematics in the Advanced Department of the Schools. Thus an opportunity is given for the latest discoveries to be disseminated among the general body of teachers.

In this way a body of public opinion is gradually being formed which regards sound mathematical teaching to boys as a national affair, closely related to the efficiency and welfare of the nation.

These reforms have come about as the result of the efforts of the Mathematical Association; and the supreme importance of the method by which they have been effected lies in the fact that they have not been imposed on teachers by some external authority, but have come as the fruit of the experience of the teachers themselves, working in close contact with the minds of the boys and making themselves acquainted with the most recent developments of their subject.

VI. THE BOY'S MATHEMATICAL DEVELOPMENT.

The boy's mathematical development depends so largely on the character of his teachers and his environment that it would be unwise

to lay down hard and fast rules as to how a given boy should be trained. On the other hand, there are certain principles which experience has shown to be sound. These principles will now be sketched in broad outline.

A boy should in the first stages be introduced to mathematics as the instrument whereby he can enumerate or measure quantitatively the concrete objects around him. He should not be confronted in the first instance with an array of definitions and abstract conceptions. Intuition should play a great part in the preliminary stages. Having used mathematical ideas and processes as instruments wherewith to solve practical problems, the boy gradually begins to see the inter-relations of the various mathematical principles and methods and to distinguish those which are mutually connected from those which are entirely independent. In fact, the subject tends inevitably to become more abstract as he proceeds and it is then possible to dwell to a somewhat greater extent upon formal definitions and root principles. To the average boy, however, the subject should throughout his whole course be taught largely from the concrete and practical standpoint. Mathematics is and should be to the average pupil predominantly a practical instrument and not an object of philosophic contemplation. These considerations suggest that a boy should study, early in his mathematical career, such subjects as Trigonometry, the Calculus, Mechanics, Physics and so on, all from the numerical and practical standpoint. This system has been found very successful. The study of general principles comes afterwards for those boys who can profit by a more abstract course.

Though the principle of adhering largely to the concrete and practical in teaching boys has not received explicit recognition among schoolmasters until the present day, an examination of history shows that the principle is not a modern discovery. On page 46 of *Britain's Heritage of Science*,* we find it recorded : "Chemistry and Botany being mainly introduced as adjuncts to medicine, it appears that science at the Universities may be said to have been confined to the application of mathematics first to Astronomy, and subsequently to other subjects, which, as they became more definite began to supply material for the exercise of mathematical skill. Experimental science for its own sake began to be taught at the Universities only in comparatively recent times. On the other hand, it is well to dispose at once of the erroneous impression that the British Universities were bodies which confined themselves to the academic discussion of abstruse subjects unrelated to the ordinary interests of the community. The Universities trained the medical men, who kept the flag of science flying in the eighteenth century, and the study of astronomy was pursued in great part for the sake of its value in finding the position of ships at sea, and in the measurement of time. The problems dealt with by mathematicians were, at first, generally suggested by practical requirements, and only gradually became detached from them. In fact, science began to be taught as a means towards a practical end."

To the boy, perhaps, there is no greater revelation of the idea of mathematical law and order in the world around him than is presented to him in the Mathematical Laboratory, which is now becoming a recognised essential of the equipment of every secondary school. Most formulae are manufactured for him and inserted in his text-book, and he is merely called upon to manipulate them, whereas in the Mathematical Laboratory he has actually to construct formulae for himself. He finds with the aid of squared paper, for example, that water descending a tube is governed by a simple formula, connecting the distance descended and the time of descent ; that the length and time of beat of

* Schuster & Shipley. Constable, 1918.

a pendulum furnish an easy example in algebra ; that a simple mathematical expression connects the power and weight of a ordinary "block and tackle."

The function of the Mathematical Laboratory is not identical with that of the Physics Laboratory, though there is much in common. The experiments of the latter are devised principally to illustrate natural phenomena, and to give practice in obtaining experimental results to a high degree of accuracy, whereas in the Mathematical Laboratory the aim ought to be to select experiments not presenting great experimental difficulty, but capable of broad effects, in order to obtain in algebraic form the law which some easily understood natural phenomenon obeys. The two aims often overlap, but their main purposes are distinct.

A common practice in schools is to teach the three older subjects of Arithmetic, Algebra, and Geometry in a leisurely way, and afterwards to hurry through some Trigonometry, Mechanics and the Calculus in the last year or year and a half of the pupils' time at school. Now it is a commonplace of pedagogy that the full grasp of a subject requires "lapse of time." If a subject be begun early, even though little time be devoted to it, the ideas peculiar to the subject will be revolved and pondered over in the pupil's mind, until he has thoroughly assimilated them. It is therefore recommended that some of the Arithmetic, Algebra, and Geometry should be left over till later, and that the time thus set free should be utilised to introduce by easy and gradual stages, early in the pupil's mathematical career, the numerical and practical study of Trigonometry, Mechanics and the Calculus.

The result of such a course of Mathematics teaching is found to be that the pupil on leaving School has studied from the concrete and practical standpoints a wide range of mathematical processes and phenomena and that he is the better equipped to apply such mathematical knowledge to the occurrences of everyday life. He is in a better position than if he had studied exclusively the abstractions of mathematical analysis, which to the vast majority of the human race are unattractive and incomprehensible.

The Historical aspect of Mathematics has never yet found its fitting place in the teaching of the schools. What one may call the Wives-of-Henry-the-Eighth type of History is gradually being abandoned, and the development of the human race in its social, intellectual and national aspects is taking its place. The development of the great mathematical discoveries might find a fitting place in such a historical scheme. Every boy can see the picturesqueness of the Greek mathematician—half merchant, half scientist—cruising about among the islands, selling his merchandise or discussing mathematical problems at his various halting-places. Boys are always interested in Cardan, who having stolen Tartaglia's Solution of the Cubic Equation announced it as his own, and who in a fit of parental rage cut off his son's ears. Were ever the morals and chivalry of the times better illustrated? The picture of the mediaeval savant wandering over Europe and nailing his problems to the doors of the various academies appeals to a boy's imagination. It is also instructive for him to realise the cost in health and strength at which great discoveries are often made, as witness the number of the great mathematicians who have died of consumption and kindred diseases from overwork and underfeeding through being too poor to purchase the prime necessities of life. Every boy ought to know something of the more human and personal side of the subject he studies. History thus acquires a new meaning for him. It would also help to vitalise the subject of mathematics if occasional lessons were given on the practical effects on civilization of mathematical

discoveries; the discoveries in geometry of the Greeks in helping forward the science of Architecture and Surveying: the effects of Kepler's astronomical discoveries on Navigation: the revolutionary character of Newton's discoveries in Optics, Mechanics, the art of Computation and so forth. So important is this aspect of teaching becoming that several books dealing in a popular style with the great mathematical and scientific discoverers and their work have of late been issued, some of which should be added to the school library.*

VII. THE TEACHER.

It has been said that the first essential of a teacher is to have something to teach. It is necessary if really satisfactory results are to be obtained that the teacher shall know his subject really well. No man can teach a subject up to the level of knowledge that he himself possesses. He must possess knowledge considerably beyond the limit to which he has to take his pupils. Furthermore it is considered desirable that the teacher shall have undergone a theoretical and practical course in the Theory and Art of Teaching so that when placed in a class-room he may have an idea of the standards to aim at and the errors to avoid. The mathematical teacher has therefore to take two distinct courses of training before his professional education is complete: (1) training in mathematical knowledge; (2) training in the science and art of teaching. Attempts have frequently been made with a view of saving time to combine the course in pedagogy with the University course in the subject of Mathematics itself. This is a mistake. It is essential that sufficient time be found, at the University, to gain a masterly knowledge of the subject itself. There should be no distraction and no wastage of effort, but complete concentration during the entire period of University study on acquiring the requisite theoretical knowledge, part of which is to be transmitted to others later on. There can be no doubt that a man will make a more satisfactory teacher in the long run if he has received a severe and adequate training in his subject and no professional training in the art of teaching, rather than if he has dissipated his energies in a desultory attendance at pedagogic lectures and classes. Time must be found for the two courses separately.

The complete course in mathematics offered by the best secondary schools in England to pupils who can take advantage of it surpasses in

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* The following may be given as examples of such books:

History of Mathematics, by Rouse Ball. (Macmillan.)
Discovery, by R. A. Gregory. (Macmillian, 1916.)
Britain's Heritage of Science, by A. Schuster & G. Shipley. (Constable, 1918.)
Pioneers of Science, by Sir Oliver Lodge. 1904.
Mathematics, by C. A. Laisant. (Thresholds of Science, Constable & Co.)
Astronomy, by Camille Flammarion. (Thresholds of Science.)
Mechanics, by C. E. Guillaume. (Thresholds of Science.)
Physics, by Félix Carré. (Thresholds of Science.)
Lectures on Ten British Mathematicians of the Nineteenth Century, by A. Macfarlane. (Wiley, 1916.)
Spinning-Tops, by Professor Perry. (Society for Promotion of Christian Knowledge.) 1890.
The First Book of Geometry, by G. C. Young and Prof. W. H. Young. (Dent, 1908.)
Mathematics, by Prof. Whitehead. (Home Univ. Library.)
The Nature of Mathematics, by P. E. B. Jourdain. (People's Books.)
Life of Kelvin (People's Books).
Faraday as a Discoverer, by G. Tyndall. (Longmans, 1870.)
A Short History of Science, by W. T. Sedgwick and W. H. Tyler. (The Macmillan Co.) 1917.
Number Stories of Long Ago, by D. E. Smith. (Ginn & Co.) 1919.
Fink's *History of Mathematics*. (Open Court Pub. Co.) 1900.

width and profundity that which can be obtained at schools in any other country. The course ranges from the very beginnings of mathematics to a fairly advanced stage in the honours course at a University. Hence there are required for English Secondary Schools not only competent teachers with a good all-round knowledge of the subject for teaching the junior forms, but also experts for the higher branches, now taught in the Advanced Departments. Masters teaching in the Advanced Departments cannot do their work really well unless they possess a superior knowledge of the subject. The French have looked further ahead and have been more sedulous in the preservation of a high standard of knowledge amongst their advanced teachers. The French Professeur de Mathématiques in the Lycée, who possesses the Agrégé qualification, has a high professional status in the eyes of the community, and public opinion encourages him to keep constantly in touch with modern mathematical developments. The French educational authorities, with that logical and practical sense which is the characteristic of the nation, provide those professeurs who take the highest work in the schools with shorter routines, so that they may have leisure for study. The Americans too are leading the way in allowing their teachers a *Wanderjahr* in which they leave their posts for a year to study at foreign universities and observe other methods and educational processes. Only by these or similar means can those teachers who teach the higher work in the secondary schools maintain a sufficiently high level of knowledge and outlook.

The Head of the Mathematical Department is frequently regarded as a man whose supreme duty is to make up time-tables. Knowledge of mathematics is a recommendation but a subsidiary qualification. Never was a greater mistake made. The Head of a Department is undoubtedly called upon to arrange time-tables for his Department, but above all things he is required so to arrange the mathematical syllabus that the pupils shall receive the best instruction in mathematics from the most approved viewpoint that the times afford. Only an expert can do this, and the fact must be faced.

The educational machinery of England cannot be considered satisfactory until conditions are such that an adequate number of high mathematicians from the universities enters the educational profession. It is not sufficient, however, to attract good mathematicians into the service ; they must be given the necessary opportunity and encouragement to remain good, and this cannot be done by overwhelming the Head of a Mathematical Side or the teacher in an Advanced Department with irrelevant duties, thereby depriving him of the necessary leisure and surplus strength wherewith to keep abreast of the times in the development of his subject. Furthermore, the Headship of a Mathematical Department is so important a post that it ought to be made an end in itself, whereas there is still continually being repeated in present-day educational life the historic tragedy of Sir Isaac Newton, who, when he had thoroughly learned his mathematical work and was ripe to go on to greater and greater triumphs, was considered to be wasting his time thereat, and received public recognition and affluence as Master of the Mint.

There exists in England a belief that if a man keeps himself thoroughly well versed in any particular subject of study he is thereby narrowing his outlook and becoming unfit for an work except in his own limited department of special study. This opinion is not supported by a study of the lives of the gifted mathematicians. Leibniz was a great diplomat in addition to being a high authority in Natural Science and Mathematics. Carnot will be known to all time for his discoveries in geometry as well as for organising the Armies of France and saving the Revolution.

Maxwell translated Faraday's electrical visions into concrete mathematical symbols besides writing inimitable epigrams and humorous poems. Kelvin was a great Natural Philosopher and at the same time an excellent man of business and affairs. Painlevé was not debarred by his knowledge of Differential Equation Theory from becoming Prime Minister of France. In fact, though it is undeniable that a few of the greater mathematicians have become so absorbed in their special studies as to become almost oblivious of the world around them and therefore unfit for the work of a teacher, it is equally undeniable that the ordinary good mathematician takes interest in many things besides his special study. It is a common experience in schools that the presence of an unusually vigorous and specially trained intellect on the staff has been of incalculable value to a school by stimulating mental activity both in the pupils and in other masters. So unlikely is it that a mathematician of the "absorbed" type should find himself on the staff of a school that the contemplation of his case may be dismissed at once. One may, however, make clearer the position and utility of that class of teacher—few in number, but of vital importance—who teach the higher portions of the subject and are responsible for directing the work of others, by considering the parallel case of the medical profession. Doctors may be divided into three main classes, (1) general practitioners, (2) consulting specialists, (3) research experts. The duties of the first class are well enough understood. The function of the third class is to devote practically all their time and energy to the discovery of new facts and methods of treatment. The second class do not have it as their main object to discover new facts, but to keep themselves well acquainted with the discoveries of the investigators of class (3), and to apply this knowledge to the case of any patient that may present himself for treatment. This ought to be the object of the specialist teacher. He has not the time or strength to make great discoveries, but he ought to be conversant with the trend of mathematical thought and to modify his teaching and syllabus accordingly. English education has in the past admitted this principle only in the case of advanced teachers of Classics, who frequently have shorter routines and other privileges in order that they may maintain themselves at the requisite level by reading and reflection. The result of this policy has been abundantly justified in the results of classical teaching in the great schools. The same principle should be applied to other subjects.

VIII. THE MATHEMATICAL SYLLABUS.

The diverse origins of mathematical education, partly commercial, partly academic, are represented by the diverse views that prevail as to the right way of teaching the subject. At the one extreme there is the school of thought known on the continent as "Perryismus" which emphasizes the utilitarian motive, and lays stress on concrete illustrations from practical life as the chief mode of presentation; at the other extreme there is the "high and dry" school, now perhaps suffering temporary eclipse, but always standing for high ideals of scholarship. In drawing up the school mathematical syllabus it will probably be wisest—and certainly safest—to steer a middle course between these two extremes of pedagogic thought.

The history of mathematics will give us some help in framing our school syllabus. Among the Egyptians, who were the pioneers of the science, mathematics took a very practical form and was the handmaid of the land-surveyor, the architect and perhaps the astronomer. The Greeks took the mathematical concepts thus reached and gave to them

a very abstract and theoretical form, which was ultimately presented to the world by Euclid in his text-book on the *Elements of Geometry*. In Europe for 2000 years, whenever an attempt was made to teach Geometry logically and deductively, the text-book adopted was in general Euclid's *Elements*, though modifications were made, principally in France, during the latter part of this period. History thus shows that the mind first forms mathematical ideas by studying practical problems. Ought we not to follow the teaching of history in forming our mathematical syllabus ? Ought we not, for example, to lay the foundations of geometry by an easy course of land surveying, in which the pupil learns to measure and calculate the area of a field, the height of a tower and so forth. They will thus acquire in a very concrete and pleasing way the idea of perpendicularity, angles, trapezia, etc. Afterwards they will be in a position to appreciate abstract geometry and to reason about the properties of triangles and parallelograms drawn on paper. Again, the meaning of an algebraic law will be better grasped if the pupil takes down his own observations in the laboratory from experiments carried out by himself, and deduces therefrom the algebraic law that they satisfy.

It is a sound pedagogic maxim that the pupil should first of all examine the simple natural phenomena around him, and test and discover the mathematical laws that they obey. At a much later stage he may be able to appreciate the minimum number of axioms on which a body of scientific knowledge is based and to view scientific facts as part of an ordered system. The study of mathematics in England by the average boy has been much retarded by compelling him to deal at the outset with definitions and abstractions, instead of introducing him speedily and directly to the concrete and letting the work become gradually more abstract as he proceeds.

IX. EXTERNAL EXAMINATIONS.

External examinations exist in the main for a twofold purpose, (1) to test whether individual candidates reach the specified standard of knowledge, and (2) to test whether the schools are training the pupils properly. In order that teachers and taught may know what is required of them, a syllabus is drawn up which is strictly adhered to by the examiner in setting his papers. Syllabuses are useful in fixing a definite goal at which to aim, but they rapidly become antiquated and bind the teacher to obsolete aims and methods. Examination syllabuses have a bad effect on the teaching of the schools unless they are continually under revision. The examination syllabus ought to be frequently overhauled by a joint-board consisting of teachers and examiners. For example, Matriculation Examinations have in the past been set mainly by University Professors without consultation with teachers. The tendency of the University Professor is to regard any subject as an ordered and systematized branch of human knowledge. He is apt to set questions of an academic and theoretical type. The mind of the boy is immature and regards mathematics as an instrument wherewith to solve any practical problem that presents itself. He is unable to face a high and exact standard of rigour. The average schoolboy cannot comprehend Euclid's doctrine of Ratio with its logical treatment of Irrational Numbers, but he finds the principles of Drawing to Scale, Map-drawing and other practical applications of Ratio quite simple and natural. Again, it is only recently that the abstract theory of the Quadratic Equation has tended to disappear from examination papers. The formal proof that a Quadratic Equation cannot have more than two roots was to many a boy very uninteresting, and at the

end of it he had little idea of what he had done. The External Examiner has refused to be convinced by the man in daily contact with the boys' minds that such hard logical reasoning is beyond them. For the boy the quadratic equation is a practical instrument for solving problems. It is a weapon to be used and not a theoretical development. Unless Examination Syllabuses are drawn up with a close acquaintance with boys' minds—what they can do and how the subject presents itself to them—the dead hand of the External Examiner is upon that branch of education, and the teacher is compelled to teach what his judgment rejects.

It is encouraging to find that some universities are co-opting teachers who, with the University Professors and Lecturers, draw up the Matriculation Syllabus, but the movement still is far from general.

X. VOCATIONAL MATHEMATICS.

The question of Vocational Mathematics deserves attention, but the experiments which are being carried out in that direction are probably too incomplete to warrant any definite conclusions. The underlying idea is this. Many teachers in districts where a special industry is predominant (*e.g.* agriculture, mining, engineering), believe that education (and in particular the mathematical syllabus) acquires vigour and sharpness of definition if the various subjects be as far as possible linked on to the phenomena of the local industry. The conception of an area to a country boy is associated with the area of a turnip field and the number of turnips it can grow. To a boy in a mining district, the idea of volume appertains to the shafts and galleries of a mine, whereas to a sea-faring lad it is connected with the volume of water in a dock or the cubic content of the hold of a ship. On the other hand, it is recognised that not all the boys in the district will enter the special industry and that the mathematical subjects or topics must to a certain extent be drawn from a wider field. The outlook of the boys would be narrowed if they did not look beyond their immediate neighbourhood. This balancing of "vocational bias" against "broad outlook" is difficult of adjustment. The headmaster of a certain rural grammar school who has gone into the subject with great thoroughness and has brought much experience to bear upon it suggests that in a district where a particular industry is predominant, about one third of the mathematics and science taught should bear upon the industry.

APPENDIX.

The articles in this appendix must be regarded as expressing purely personal views and not the considered opinions of the Mathematical Association Teaching Committee. Their value lies in their being the opinions of teachers of distinction and experience.

CORRELATION OF MATHEMATICS AND SCIENCE.

By a former Head of a Mathematical Department.

It is a commonplace of present-day educational theory that a school curriculum ought not to be divided into "watertight compartments." On the other hand, a system of compartments of some sort is the natural and normal form of present-day school organisation. Such a system not only facilitates the employment of a specialised staff, the advantages

of which are universally admitted, but also ensures to the pupil variety of treatment and variety of interest with corresponding breadth of outlook.

A system of no compartments at all is apt to result in all-round mediocrity, rendered all the more dangerous by the fact that its deficiencies are not realised by those who have attained it. "Water-tight compartments," on the other hand, tend to result in skill and knowledge without the power to apply either outside the narrow limits within which they have been acquired. Both systems fail in different ways to develop that power over nature—using "nature" in its very widest sense—which we call civilisation and which we look for, in greater or less degree according to the individual, as the outcome of education.

Between these two extremes of disadvantage there obviously lies some system of maximum advantage the exact nature of which will depend on the character and natural relations of the subjects concerned as well as on the circumstances of the individual schools. Mathematics, almost more than any other subject of study, needs to have its relationship to neighbouring subjects kept constantly in the forefront. There are few persons, either juvenile or adult, to whom purely mathematical study is an interest in itself, while at the same time there are few departments of human thought in which some amount of application of mathematical processes is not necessary. Consequently, if we are to maintain that interest which is essential to the profitable study of mathematics itself, and if we are to secure that the study of other subjects shall by the application of mathematical methods become truly effective and thorough, it is important that the lines of intercommunication between mathematics and its neighbours shall be both numerous and wide.

Without some regular provision for transference of thought from one compartment to another, it is the commonest experience of the schoolmaster that the existence of such transference cannot be taken for granted. The boy who could work his sums only in apples is merely an extreme instance of an everyday phenomenon. How often do we hear recriminations in the Common Room that a class that is supposed to have done decimals is unable to apply them to measurements taken in the laboratory, or that a class that is supposed to have done "densities" in the laboratory knows nothing about them when it comes to working sums? We hear the same kind of recrimination in the case of transfer of a pupil from one educational institution to another, and in its most aggravated form from the heads of firms who complain, when the boys or girls leave school, that the schools have taught them nothing. Doubtless such complaints, although at times unreasonable, are for the most part well founded. It is all a matter of association and environment. What a youth can accomplish under one set of associations he may fail to accomplish when the environment is completely changed. It behoves us then as pedagogues to take steps to ensure that the associations of any piece of work shall be sufficiently varied, that the work itself, both in its foundation and in the superstructure resting upon it, shall be brought, almost forced, into relation with the larger activities of which it forms a part. Not, of course, by way of irrelevant digression, as when the instructor of woodwork communicates to his wondering class his meagre stock of botanical information, or when the botany mistress digresses on the politics of the Primrose League. The shoemaker must stick to his last. But the shoes he makes must stand the wear and tear of everyday life, and he will not refrain from a judicious use of iron merely because his trade happens to be more intimately connected with leather.

There are, at any rate, two applications of mathematical processes

which it has always been customary to associate closely with the purely mathematical teaching. These are Geometry, or the application to space relations, and Monetary Arithmetic. To some extent the applications themselves have suffered by the divorce from concrete experience which this association has often entailed. At the present day this particular trouble is well on the way to elimination by the change towards a greater respect for the concrete that is coming over mathematical teaching in general. The question, however, arises as to what other applications ought to be intimately associated with mathematical teaching besides Geometry and commercial transactions, which nowadays represent a relatively narrow field of mathematical application. Since the time of Galileo the field has been ever widening until now it touches all branches of Science, and embraces virtually the whole of Physics and a very large part of Chemistry. Now that we have reached a stage when elementary Physics and Chemistry form an essential part of our school curriculum there does not appear to be any fundamental reason why the relationship of Physics at least to Mathematics in our schools should not be of the same nature as that of Geometry or any other branch of Applied Mathematics.

Of the actual obstacles to combining the two subjects the first is the difficulty of staff. In the present generation many of the best mathematical teachers have little training in or experience of experimental work. Certainly Physics is much more exacting in this respect than Geometry or Money Sums. Therein lies a great part of its value. Secondly, experimental work in Physics requires so much apparatus and elbow-room that it can be dealt with successfully only in a laboratory. In most schools laboratory space is too precious at the present time to be occupied for other than purely experimental work during any considerable portion of the teaching day, so that it is only in exceptional cases that a course of Mathematics including Physics is likely to come into existence under present conditions. The advantages, however, of such a course are obvious. It would, to begin with, knit up the association between theory and practice that to-day is often so far to seek. With the fundamental assumptions of the reasoning securely based on experiment, the pupil would go forward to the conclusion with that confidence which only familiarity with material things can give. An experiment would no longer tend to be, as it is to be feared at present it often is, a mere glimpse of some physical phenomenon, neither prepared for by sound mathematical reasoning nor followed by such reasoning to its logical conclusion, and made part of the mind by the force of numerous examples. Mathematics, instead of being limited in application to space and money, would be seen to be applicable to a great variety of phenomena. The teacher's hunt for artificial problems would cease. And "x" would no longer be a toy key for solving puzzles, but an implement to help to wrest from Nature the secrets of the unknown.

Although, as indicated, such a course may be at present to a large extent an unrealisable ideal, yet so far as staffing is concerned the obstacles are in process of being overcome. The number of teachers capable of combining Mathematics and Physics is rapidly increasing, and many who have not had the good fortune of a training in both Departments are not too old to acquire the necessary interest and skill. Where a school can be staffed so that the Science and Mathematics are shared by the same members much useful work of a joint nature can be done. Where such staffing is not possible it ought at least to be imperative that the teachers of Mathematics and of Science should be thoroughly familiar with the details of the syllabus in both subjects. The syllabuses should indeed as far as possible be planned with a view

to the closest co-operation, and the teacher of a class in the one subject should be thoroughly conversant with the progress of the class in the other. With co-operation of this kind there is full opportunity for continual cross-reference between the two most intimately connected branches of the school work. The pupils become accustomed to applying with one teacher, and in a different environment, what they have learnt with another. The unhealthy isolation of the separate branches is broken down, and education ceases to be a repetition of specified tasks amid artificial surroundings, becoming instead the acquisition of power to direct the mental processes, generated within the class-room or the laboratory, to things without that are of wider and more general interest. It becomes the cultivation of wisdom as distinguished from the mere pursuit of learning.

The intimate connection between Science and Mathematics begins as soon as systematic study of Science is undertaken; that is when the pupil is about twelve years of age. Thereafter it is doubtful if the connection ever really ceases. Certainly, as far as present-day school courses are concerned, progress in the one subject is greatly dependent on progress *pari passu* in the other. This intimate connection is fully recognised by the Board of Education in their recent proposals for Advanced Courses in Secondary Schools, but it is unfortunate that the principle of a Science and Mathematical group has not received the same recognition at the hands of the Universities of Oxford and Cambridge. These provide in their Higher School Certificate Examinations only for separate groups in Mathematics and in Science, thereby encouraging that one-sided development and narrowness of outlook which the system of entrance scholarships at the older Universities has already done so much to promote. Whatever may be said in favour of more restricted specialisation at the University, it is certain that during the school period the young mathematician should be brought up to realise the bearing of his study on the problems of nature, while the young scientist should be put in possession of the intellectual tools he will certainly require.

THE RELATIONS OF MATHEMATICS AND MANUAL TRAINING.

BY T. S. USHERWOOD.

In his ***Ethical Principles Underlying Education,*** Professor Dewey says: "The moment mathematical study is severed from the place which it occupies with reference to social life, it becomes unduly abstract —even from the intellectual side." The mathematical problems met in the workshop—whether it be for Science, Geography or Manual Training—afford the most natural introduction to the formal study of mathematics, because it is in these workshops that desire for self-expression and interest in construction are most closely linked with manipulation of numbers, realisation of the value of symbols, and acquaintance with the facts of geometry: in addition they supply the most convincing tests of mastery. The problems met are definitely within the experience of the imagination of the pupil, and the desire to find some means of dealing adequately with them is a prime motive to investigation.

In but few schools—so far as can be discovered—is there any deliberate attempt at organising a scheme in which full advantage of workshop facilities is taken: but since it is only in very elementary work that there is close contact between mathematics and handwork (if we except the case of older boys taking applied science and engineering) it ought not to be difficult to arrange a series of manual exercises which

should at the same time have a definite mathematical aim, if not a deliberate bias. Such a series should, however, not be allowed to become stereotyped: there must be scope for inventiveness and initiative; the desire for self-expression must be respected; but the construction of any useful article is almost impossible without mathematical knowledge, and it is a matter of co-ordination to get the best results both mathematically and practically. In most Secondary schools no direct use is made of the Manual Training School in the mathematical class-rooms: the reason probably is that a definite and reasoned mathematical scheme was bearing fruit long before a workshop was contemplated, and the economical aspects of correlation have not been considered or realised.

Rationalisation of elementary work in mathematics is necessary not only to prevent its deteriorating into unintelligent and mechanical work, narrow in scope and restricted in application, but also, in these days of the overcrowded curriculum, to avoid overlapping and waste of valuable time and opportunities. While much has been done and some advantage taken of the way in which number and geometrical concepts lend themselves to concrete illustration, it is too often forgotten by teachers that the mere clothing of a problem in "practical" terms does not make that problem really practical, and cannot arouse the same kind of interest or the same keenness as is felt when the solution of a problem must be attempted for adequate self-expression. It may also be urged, in fact, that formal treatment precedes "practical" needs, instead of arising from the latter as a desirable if not inevitable piece of research, as it should.

What is really necessary is co-ordination between class-room and workshop. The latter affords unrivalled opportunities for simple measurement and calculation, and for those spatial experiences which form the foundation of formal geometry; at the same time inaccuracy brings its own punishment. Further, the preparation of working drawings leads through the use of scales to ideas of ratio and proportion, and, in its higher developments, to the use of the important weapon forged by Monge and indispensable to the engineer.

THE TRAINING OF THE TEACHER.

By Professor T. Percy Nunn, M.A., D.Sc.

Current opinions on the value of training for the teacher cover the widest possible range. There are stalwarts who believe that if a teacher knows "pedagogy" he need know little else; there are sceptics who think that nothing matters except knowledge of the subject to be taught, and dismiss the pretensions of the training college as mischievous humbug. There is something to be said for both these extreme views. Just as an ignorant doctor may often, by a "good bedside manner," stimulate the *vis medicatrix naturae* which really cures his patient, so a teacher ignorant of his subject but cunning in his art, may often awaken in the pupil the inner energies that are the ultimate source of educational progress. And just as a ripe orange must yield its juice when squeezed, so the poorest of teachers, if he has learning, must deliver it to pupils who are eager enough to suck it from him.

To apologise for these antithetical opinions is to expose their weakness, and to show that neither can stand unless supplemented by the other. On the one hand, it is futile for a teacher to evoke an "interest" which he cannot feed with the nutriment it craves; on the other hand, learning in a teacher is a vain possession if he cannot persuade his pupils, first, that it is worth having, and secondly, that he can give

it to them. Nothing is to be gained by arguing about the relative importance of conditions where both are indispensable. It is plain that the teacher of mathematics must both learn his subjects and learn how to teach them; the only questions worth discussing are questions of ways and means.

Upon one of these questions there is now general agreement among competent judges. In the making of the teacher, the "academic" stage—learning the subject—should be completed before the "professional" stage—learning how to teach it—is seriously begun. There are several reasons for this policy, but the most cogent is that the student's mind ought not to be distracted between two interests of which each should be large enough and active enough to drive all other interests into the background. During his undergraduate years, therefore, the future teacher should be simply a student among other students, conscious that he has a vocation awaiting him, but, in the meantime, throwing himself whole-heartedly into the purely intellectual activities which the pursuit of his subject entails. It follows that the scope and character of his mathematical studies should not be limited by his (supposed) professional needs. This is not to admit that existing University courses in mathematics are plenarily inspired, nor to deny that many of them would be much more useful to teachers if they were somewhat drastically changed. But such changes are to be pressed for on the broad ground that what the teacher needs above all other students is a course which represents adequately the essential genius of mathematics, and that, as things are, he cannot always get it.

Having made this position clear we may pass on to consider, in its broad lines, the student's professional training. Here we must bear in mind that we are concerned not merely to turn out a competent craftsman, but to form a young man or a young woman into an enlightened member of a body that has an enormous responsibility for the well-being of a nation. Before the young graduate's outlook has suffered the narrowing so often produced by professional routine, while he is still warm with the generous and universal spirit that University life should have awakened in him, he should be led to inquire into the meaning of education and to understand something of its significance in relation to the many-sided business of life. He should learn how much wider education is than mere teaching, and should gain inspiration and right direction from those who have reflected most deeply upon it. And he should have his bias as a specialist corrected by observing how all the major subjects of the curriculum answer to deep-seated needs of the human spirit and represent essential currents of the great stream of movement called civilisation.

Nothing could be, in the long run, more unwise than to exclude from the professional course those elements of breadth and liberality in which, as everyone agrees, the chief virtue of academic studies resides. Nevertheless the course will miss its point unless it also includes an adequate training in the teaching-craft that belongs to the student's special subject. It is not sufficient for him to learn the common arts of exposition and class-management; he must learn what forms those arts assume when applied, for example, in the field of mathematical teaching. The first thing he should discover here is that the art of teaching does not consist in the mastery of a number of tricks by which young minds may be persuaded to accept a mass of ready-made knowledge; but that it is a process whose aim is to guide the pupils' mental activities along a path of development in which he repeats and makes his own some of the great historic achievements of the science. In other words, he must learn that to teach mathematics is not merely to seize the pupil of certain knowledge in arithmetic, algebra, geometry and the rest, but to

make him, in the greatest measure possible, an active intellectual adventurer in the realms of number and space, following up the traces of the great masters of mathematical thought and catching something of their spirit and outlook. To make the best use of this discovery the student must work again over the familiar field of elementary mathematics, studying it, this time, not in the naïve attitude of a learner, but as a critic interested in finding out whence mathematical thought springs, how it develops and whither it leads. In this part of his training he should be taught to question the accepted values, and to inquire in a critical spirit what parts of the traditional curriculum are really vital and what parts have only a conventional value. He should also be led to study the reactions to mathematical teaching of minds differing in age and natural bent: to observe the appeal, in varying circumstances of the intrinsic beauty, the utility and the logical unity of mathematical truths.

The question who should teach the teacher his trade must be regarded as still unsettled. Some hold that he should learn his craft entirely in a school under the eye of a master-craftsman; others that he should spend his year of training under the direct influence and guidance of a training college or pedagogical department of a University. The former plan has the obvious merit of keeping the young teacher constantly in touch with the actual conditions of his profession, but it has serious drawbacks. A school can rarely be a place where a student can gain that philosophic outlook upon education as a whole upon which we have insisted, or study effectively the more recent contributions of psychology to the business of teaching. Moreover, there is only a very small number of highly competent teachers of mathematics who can spare the time and have the special interest needed both to supervise adequately the aspirant's technical progress and to guide his studies in what we have spoken of as the criticism of his subject. The ideal solution of the problem of training would seem to involve an institution of University rank—a centre of educational criticism and inspiration, and a clearing-house of educational ideas—with schools working in such close relations with it as to be true organs of its spirit. It is, perhaps, not extravagant to hope that as our newer Universities gather strength, such institutions will grow up in them, and become, so to speak, the centres of consciousness of the teaching profession, each in its own province. Having earned the necessary prestige, they could not only supervise the apprenticeship of young teachers, but also perform the functions of the military "staff college" as places to which eminent teachers could be invited to withdraw for a while from the busy routine of school life in order to give their colleagues in the schools of the province the benefit of their experience and special skill, and to keep them well in the stream of progress. For this, too, is an aspect of the question of training of no less importance than the one which the term ordinarily suggests.

THE TEACHER AND RESEARCH.

By Professor E. T. Whittaker, Sc.D., F.R.S.

The importance of research has been more fully recognised during the last thirty years. A generation ago the majority even of the mathematical Fellows at Cambridge made no contributions to the progress of the subject; to-day perhaps the most important factor in any appointment to a teaching office of University rank is the candidate's record of original work. It is now not unusual in adver-

tisements of vacancies to see "ample time and opportunity for research" included among the attractions of the position.

The schoolmaster who wishes to undertake original investigations has more obstacles to overcome than the College lecturer. But there can be no doubt of the existence of a widespread interest in this direction. Some time ago I was asked to speak to a gathering of mathematical teachers in the Scottish secondary schools. "What is wanted?" I said—"Something about the teaching of mathematics?" "On no account," was the reply. "We have heard about the teaching of mathematics *ad nauseum*: tell us about the new discoveries in the subject."

The existence of this interest is, in my opinion, one of the things needed to save the profession of mathematical teaching from a grave peril: the peril, in fact, of disruption. For if a state of things came to pass in which the University teachers were all interested in Research, while the School teachers were interested in nothing beyond Pedagogy, it is evident that the two branches of the profession would soon become two distinct professions with no bond of sympathy or union. The consequences of such a separation would be from every point of view regrettable: on the one hand, the possibility of translation to University work would be denied to the Schoolmaster: and on the other hand the schools would be deprived of that touch with the Universities which in the past has often been created by the appointment of a University Assistant to a mastership.

What, then, are the hindrances in the way of the School teacher who is interested in research, and how can they be removed?

The chief one is undoubtedly the difficulty of choosing a subject likely to yield results of value to the investigator, and of finding out what has been published already regarding it. The literature of mathematics is so vast that it is almost impossible for a beginner in research to make a satisfactory start without some help from a man of wider knowledge and experience. The modern post-graduate school attached to a University Department of Mathematics is designed to meet this need, so far as the whole-time research student is concerned. But how can something of the kind be made available for the schoolmaster?

In my own Department in Edinburgh courses of post-graduate lectures, leading to topics suitable for original investigations, are given during term-time twice a week at four o'clock, when the Edinburgh schools have closed for the day. Several teachers of mathematics in the secondary schools, etc., of the city have taken advantage of this arrangement, and a recent D.Sc. degree in Mathematics was awarded to a member of the class who has never interrupted his daily work as a teacher.

A system of this type might, I think, be developed to a considerable extent. It is only applicable in University towns; but the number of University towns is now happily large, and the teacher who wishes for something of the kind would endeavour to obtain an appointment in one of them.

A second suggestion is the Colloquium, which is a gathering held at some University centre during the Long Vacation, when three or four short courses of post-graduate lectures are given by specialists. The experience of the Edinburgh Mathematical Colloquium in 1913 and again in 1914 was encouraging: but the courses were too brief to lead fully into research topics, and the membership was too great to make much personal guidance possible. The organisers feared that if they advertised a longer Colloquium, nobody would come to it: but it is perhaps true that really more would be accomplished if ten came for

a fortnight than if a hundred came for a week. Obviously, however, the finance of a colloquium of this type would present difficulties.

Lastly, another plan that may be put forward is the institution of a *Bulletin for the promotion of Mathematical Research*, containing the substance of post-graduate lectures (advanced theories that have never been codified into text-books, described without proofs but with references to the original memoirs and with attention called to further possibilities of development and indications of methods which might be tried in the first instance). For the mathematicians who are isolated in the remoter stations of the Empire, this seems to be the only possible solution of the problem: it was a Professor in a Mission College in India who first suggested it to me.

So far as the men not yet out in the world are concerned, the best thing of all would be that the year of "training" in pedagogy, which is customary in Scotland, and which (so far as I am able to form an opinion) is of very little value, should be abandoned, at any rate in the case of the abler graduates in mathematics and science, and replaced by a year's whole-time initiation into research in a regular post-graduate department of a University. This would give the teacher the equipment necessary for continuing original investigation, wherever his later fortunes might lead him.

GLEANINGS FAR AND NEAR.

32. Dr. Barrow's Words prefixed to his "Apollonius." God always acts Geometrically. How great a Geometrician art Thou, O Lord! For while this Science has no Bound: while there is for ever room for the Discovery of new Theorems, even by Human Faculties, Thou art acquainted with them all at one View, without any train of Consequences, without any wearisome Application of Demonstrations. In other Arts and Sciences our Understanding is able to do almost nothing; and, like the Imagination of Brutes, seems only to dream of some uncertain Propositions: Whence it is that in so many men are almost so many minds. But in these Geometrical Theorems all Men are agreed. In these the Human Faculties appear to have some real Abilities, and those Great, Wonderful and Amazing. For those Faculties which seem of almost no force in other Matters, in this Science appear to be Efficacious, Powerful and Successful, ... Thee therefore do I take occasion to Love and Rejoice in, and Admire; and to pant after that Day, with the Earnest Breathings of my Soul, when Thou shalt be pleased, out of Thy Bounty, out of Thy Immense and Sacred Benignity, to grant me the favour to perceive, and that with a pure Mind, and clear Vision not only those Truths, but those also which are more numerous, and more important; and all this without that continual and painful Application of the Imagination which we discover these withal. . . . .

33. I think Legendre has very obviously fallen into this misconception (*Fonctions Elliptiques*, vol. ii. p. 429), but it has led him to no false results. Indeed it is obvious that confounding "A may be written for B" with "A is equal to B," though it must affect the logic, may not affect the result, of a process.

The greatest writers on mathematical subjects have a genius which saves them from their own slips, and guides them to true results through inaccurate expression, and sometimes through absolute error (see that of Legendre, page 595). But their humbler followers must not permit themselves such license, and those above all who write for students must correct that as an error of reasoning, which, in the guide they follow, was little more than an error of the pen.—De Morgan, *Diff. and Integral Calculus*, n.p., 619.

MATHEMATICAL NOTE.

536. [D. 6. c.] *The series for* $\sin x$ *and* $\cos x$.

I. By actually multiplying the absolutely convergent series for e^{z_1} and e^{z_2}, we prove that
$$e^{z_1}\times e^{z_2}=e^{z_1+z_2},$$
where z_1 and z_2 may have any finite values, real or complex.

Putting $z_1=ix$, $z_2=-ix$, where $i=\sqrt{-1}$ and x is real, we get
$$e^{ix}\times e^{-ix}=e^0=1.$$

Now
$$e^{ix}=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right)$$
$$+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right). \qquad (1)$$

Since the product of conjugate complex numbers is the square of the modulus of either, we have
$$|e^{ix}|=\sqrt{e^{ix}\times e^{-ix}}=1;$$
$$\therefore\ e^{ix}=\cos X+i\sin X, \qquad (2)$$
where X is some quantity depending on x.

$\therefore$ in the Argand diagram e^{ix} is represented by a point P on the circle $|z|=1$, such that OP makes an angle X with the real axis Ox.

II. If
$$e^{ix_1}=\cos X_1+i\sin X_1$$
and
$$e^{ix_2}=\cos X_2+i\sin X_2;$$
$$\therefore\ e^{i(x_1+x_2)}=e^{ix_1}\times e^{ix_2}$$
$$=(\cos X_1+i\sin X_1)(\cos X_2+i\sin X_2)$$
$$=(\cos X_1\cos X_2-\sin X_1\sin X_2)$$
$$+i(\sin X_1\cos X_2+\cos X_1\sin X_2)$$
$$=\cos(X_1+X_2)+i\sin(X_1+X_2); \qquad (3)$$

$\therefore$ if in the Argand diagram,

P_1 represents e^{ix_1}, P_2 represents e^{ix_2}, and Q represents $e^{i(x_1+x_2)}$.

Then
$$Q\hat{O}x=P_1\hat{O}x+P_2\hat{O}x=X_1+X_2$$
and
$$Q\hat{O}P_1=P_2\hat{O}x=X_2.$$

III. If
$$e^{ix}=\cos X+i\sin X, \qquad (4)$$
and the *principal value* of the amplitude is taken (*i.e.* $-\pi<X\leq\pi$),

then
$$e^{i\frac{x}{2}}=\cos\frac{X}{2}+i\sin\frac{X}{2},$$
since each is that square root of the corresponding side of (4) which reduces to $+1$ when x (and therefore X) reduces to the value 0.

For from (1),
$$\cos X=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots, \qquad (5)$$
$$\sin X=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots; \qquad (6)$$
and therefore if x is small, $\cos X$ is nearly equal to 1, and $\sin x$ is nearly equal to 0 and is of the same sign as x;

$\therefore$ if x is small, so is X, and they are both of the same sign.

IV. Now take x to be any positive quantity, and let

$$x = 2h_1 + 2h_2 + \dots + 2h_n,$$

where every h is small and positive, and n is a large number.

Let $$e^{ix} = \cos X + i \sin X,$$

$$e^{ih_r} = \cos H_r + i \sin H_r, \quad \text{.................................(7)}$$

where H_r is the principal value of the amplitude of e^{ih_r}, and is consequently a small positive angle.

Consider the points in the Argand diagram representing

$$1,\ e^{2ih_1},\ e^{2i(h_1+h_2)},\ \dots,\ e^{2i(h_1+h_2+\dots+h_r)},\ \dots,\ e^{2i(h_1+h_2+\dots+h_n)} (= e^{ix}).$$

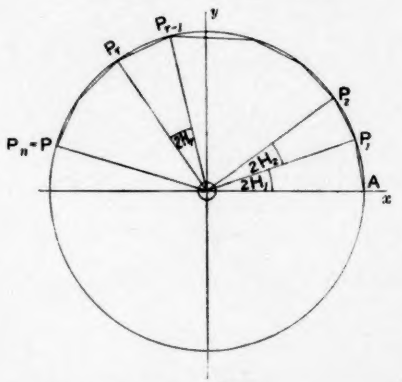
Call them $A,\ P_1,\ P_2,\ \dots,\ P_r,\ \dots,\ P_n (= P)$.

They all lie on the circle $|z| = 1$, and

$$P_r \hat{O} P_{r-1} = 2H_r,$$

$$P\hat{O}A = 2(H_1 + H_2 + \dots + H_n).$$

Also $\cos X = \cos P\hat{O}A, \quad \sin X = \sin P\hat{O}A.$



The points occur in the order $A,\ P_1,\ P_2,\ \dots,\ P_{r-1}P_r \dots P_n$ as we go round the circle from A in the direction opposite to the clock (see figure).

Join up the successive points, thus forming part of a polygon inscribed in the arc AP.

Length of the chord $P_rP_{r-1} = 2 \sin H_r$;

$\therefore$ length of the broken line $AP_1P_2 \dots P_{r-1}P_r \dots P_n$

$$= \sum_{r=1}^{n} 2 \sin H_r$$

$$= 2 \sum_{r=1}^{n} \sin H_r,$$

and by (6) and (7),

$$= 2\left\{ \sum_{r=1}^{n} \left(h_r - \frac{h_r^3}{3!} + \frac{h_r^5}{5!} - \dots \right) \right\}$$

$$= 2 \sum_{r=1}^{n} h_r - \frac{2}{3!} \sum_{r=1}^{n} h_r^3 + \dots + (-1)^s \frac{2}{(2s+1)!} \sum_{r=1}^{n} h_r^{2s+1} + \dots . \quad \text{............(8)}$$

But $$2\sum_{r=1}^{n} h_r = x \quad \text{...(9)}$$

and $$2\sum_{r=1}^{n} h_r^{2s+1} \leqq k^{2s} \cdot 2\sum_{r=1}^{n} h_r$$

$$\leqq k^{2s} \cdot x, \quad \text{...(10)}$$

where k is the largest of the h's.

Now let n increase indefinitely, and at the same time let all the h's become vanishingly small.

Then k^{2s} vanishes if $2s>0$;

$$\therefore \Sigma h_r^{2s+1} \text{ vanishes if } s>0.$$

It is then easily seen that the right-hand side of (8) tends to the value x.

The broken line $AP_1P_2 \ldots P_n$ approaches the arc AP;

$\therefore$ in the limit, $\qquad$ arc $AP = x$;

and, since radius of circle is unity,

$$P\hat{O}A = x \text{ radians};$$

$$\therefore \cos X = \cos x,$$

$$\sin X = \sin x,$$

$$\text{i.e.} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots,$$

or $e^{ix} = \cos x + i \sin x$ if x is positive.

If x is negative and equal to $-y$,

$$e^{ix} = e^{-iy} = \cos y - i \sin y$$
$$= \cos(-y) + i \sin(-y)$$
$$= \cos x + i \sin x,$$

and the series still hold. F. JACKSON.

THE LIBRARY.

CHANGE OF ADDRESS.

THE Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

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